30TH INDIAN NATIONAL MATHEMATICAL OLYMPIAD-2015

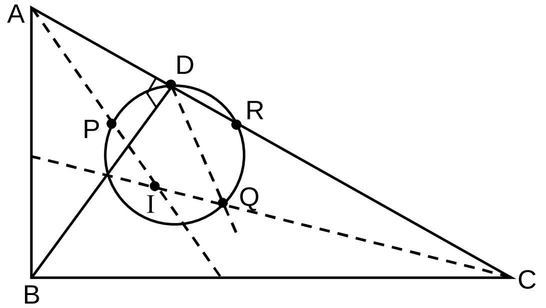
Time: 4 hours **February 01, 2015**

Instruction :

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions. Maximum marks : 100.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let ABC be a right-angled triangle with $\angle B = 90°$. Let BD be the altitude from B on to AC. Let P, Q and I be the incentres of triangles ABD, CBD and ABC respectively. Show that the circumcentre of the triangle PIQ lies on the hypotenuse AC.

Sol.



In ΔAIC

$\Delta PIQ = \pi - (\angle IAD + \angle DCI)$

$= \pi - \left(\frac{A}{2} + \frac{C}{2}\right)$

$= \pi - \left(\frac{\pi}{4}\right)$

$= \frac{3\pi}{4}$

Now Let circum-circle of ΔDPQ intersect AC again at R then $\angle PRQ = \frac{\pi}{2}$

(as $\angle PDQ = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$)

Now clearly $\angle CDQ = \angle RDQ = \frac{\pi}{4}$

$= \angle RPQ$ (as P, D, R, Q are concyclic points)

$\Rightarrow \angle RPQ = \frac{\pi}{4} \Rightarrow RP = RQ$(as ΔRPQ is right angled Δ)

Now if draw a circle with centre as R and radius as RP(=RQ) then $\angle$ made by PQ at centre $= \frac{\pi}{2}$

$\Rightarrow \angle$made by PQ at major segment $= \frac{\pi}{4} \Rightarrow \angle$ made by PQ at minor segment $= \frac{3\pi}{4} = \angle PIQ$

$\Rightarrow$ hence I lies on this circle with centre at R.

and hence clearly circumcentre of ΔPIQ (i.e. R) lies on AC.

2. For any antural numbe n > 1, write the infinite decimal expansion of 1/n (for example, we write $1/2 = 0.4\overline{9}$ as its infinite decimal expansion, not 0.5). Determine the4 length of the non-periodic part of the (infinite) decimal expansion of 1/n.

Sol. $\frac{1}{n} = 0.a_1\, a_2\, a_3 \,.....\, a_k\, b_1\, b_2\, b_3 \,.......\, b_r$

r is length of repeating part

$\frac{1}{n} = \frac{a_1a_2....a_k}{10^k} + \frac{b_1b_2....b_r}{10^{k+r}} \qquad \left(1+\frac{1}{10^r}+\left(\frac{1}{10^r}\right)^2+.......\infty\right)$

$= \frac{a_1a_2....a_k}{10^k} + \frac{b_1b_2....b_r}{10^k}\left(\frac{1}{10^r-1}\right)$

$= \frac{(a_1a_2........a_k)(10^r-1)+b_1b_2..........b_r}{(10^r-1)(10^k)}$

hence $10^k = 2^k.5^k \Rightarrow$ k is higher of the two exponents of 2 5 in prime factorization of n.

$\Rightarrow$ k = maximum {exponent of 2, exponent of 5}

3. Find all real functions f from $R \to R$ satisfying the relation

$$f(x^2 + yf(x)) = xf(x + y)$$

Sol. $f(x^2 + yf(x)) = x.f(x + y)$ ....(1)

$f(yf(0)) = 0 \Rightarrow f(0) = 0$

for y = 0, x = – x

$f(x^2) = -xf(-x)$

for y = 0 $f(x^2) = xf(x)$

$\Rightarrow xf(-x) = -xf(x)$

$\Rightarrow f(-x) = -f(x)$

$\Rightarrow$ function can be analysed only for x > 0

at x = 1 f(1 + yf(1)) = f(1 + y)

if $f(1) \neq 1$

then if f(1) = k > 1 then put $y = \frac{y}{k}$

$f(1 + y) = f\left(1+\frac{y}{k}\right)$

$= f\left(1+\frac{y}{k^2}\right)$

$= f\left(1+\frac{y}{k^n}\right)$

now as $n \to \infty$ f(1 + y) = f(1)

$\Rightarrow$ f(y) = k

from (1) $k = kx \ \forall x$

$\Rightarrow$ k = 0

f(x) = 0

similarly for k < 1

we can prove f(x) = 0

$f(x^2 + yf(x)) = x f(x+y)$

put y = 0

$f(x^2) = xf(x)$ .... (iii)

& put y = – x

$f(x^2 - xf(x)) = 0$ ($\because$ f(0) = 0)

so $x^2 - xf(x) = 0$ .... (iv)

from (iii) & (iv) f(x) = x

Hence combining all we have

either f(x) = 0

Or f(x) = x

4. There are four basket-ball players A, B, C, D. Initially, the ball is with A. The ball is always passed from one person to a different person. In how many ways can the ball come back to A after seven passes ? (For example $A \to C \to B \to D \to A \to B \to C \to A$ and $A \to D \to A \to D \to C \to A \to B \to A$ are two ways in which the ball can come back to A after seven passes.)

Sol. Let P_i denotes i^{th} pass then

$$1 \xrightarrow{P_1} 2 \xrightarrow{P_2} 3 \xrightarrow{P_3} 4 \xrightarrow{P_4} 5 \xrightarrow{P_5} 6 \xrightarrow{P_6} 7 \xrightarrow{P_7} 8$$

C–1

If in between A gets 2 passes

(1) A × A × A × × A $\to$ 3.1.3.1.3.2.1 = 54

(2) A × A × × A × A $\to$ 3.1.32.1.3.1 = 54

(3) A × × A × A × A $\to$ 3.2.1.3.1.3.1 = 54

C-2

If in between A gets 1 pass

(4) $\to$ A × A × × × × A $\to$ 3.1.3.2.2.2.1 = 72

(5) $\to$ A × × A × × × A $\to$ 3.2.1.3.2.2.1 = 72

(6) $\to$ A × × × A × × A $\to$ 3.2.2.1.3.2.1 = 72

(7) $\to$ A × × × × A × A $\to$ 3.2.2.2.1.3.1 = 72

C-3

If in between A gets no. pass

(8) $\to$ A × × × × × × A $\to$ 3.2.2.2.2.2.1 = 96

Hence total number of ways = 3.54 + 4.72 + 96

= 546

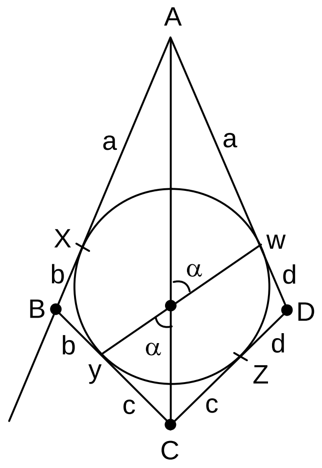
5. Let ABCD be a convex quadrilateral. Let the diagonals AC and BD intersect in P. Let PE, PF, PG and PH be the altitudes from P on to the sides AB, BC, CD and DA respectively. Show that ABCD has an incircle if and only if

$$\frac{1}{PE} + \frac{1}{PG} = \frac{1}{PF} + \frac{1}{PH}$$

Sol. Let AC and YZ intersect at Q then $\angle QYC = \angle QWD$ = Q(say)

(As $\angle$ made by tangent at apoint is same as $\angle$ made by chord through point in alternate segment)

Now In ΔAQW



$\frac{AQ}{AW} = \frac{\sin(\pi-\theta)}{\sin(\alpha)}$

$= \frac{\sin(\theta)}{\sin(\alpha)}$ .... (1)

and In ΔQCY

$\frac{CQ}{CY} = \frac{\sin(\theta)}{\sin(\alpha)}$ .... (2)

from (1) and (2) $\frac{AQ}{AW} = \frac{CQ}{CY} \Rightarrow \frac{AQ}{CQ} = \frac{AW}{CY}$

$\Rightarrow \frac{AQ}{CQ} = \frac{AW}{CY} = \frac{a}{c}$

Now if XZ intersect AC at Q' then

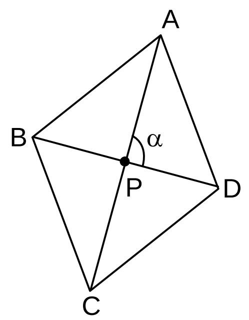
$\frac{AQ'}{CQ'} = \frac{a}{c} = \frac{AQ}{CQ} \Rightarrow Q \equiv Q'$

$\Rightarrow$ AC, XZ and YW are concurrent similarly BD, XZ and YW are concurrent

Hence AC, BD, YW and XZ are concurrent

$\Rightarrow Q \equiv P$

Let Area of MNJ = [MNJ]



also

$AP = \frac{a}{a+c} AC$

$PD = \frac{d}{b+d} BD$

$\Rightarrow$ [APD] = $\frac{1}{2}$ AP.PD. sin(α)

= $\frac{adAC.BD}{(a+c)(b+d)}$. sin(α)

= λad = $\frac{1}{2}$. AD . PH = $\frac{1}{2}$ (a + d)PH

$\Rightarrow \frac{1}{PH} = \frac{1}{2\lambda}\left(\frac{a+d}{ad}\right) = \frac{1}{2\lambda}\left(\frac{1}{a}+\frac{1}{d}\right)$

similarly $\frac{1}{PE} = \frac{1}{2\lambda}\left(\frac{1}{a}+\frac{1}{b}\right)$

$\frac{1}{PF} = \frac{1}{2\lambda}\left(\frac{1}{b}+\frac{1}{c}\right)$

$\frac{1}{PG} = \frac{1}{2\lambda}\left(\frac{1}{c}+\frac{1}{d}\right)$

$\Rightarrow \frac{1}{PE} + \frac{1}{PG} = \frac{1}{2\lambda}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) = \frac{1}{PF} + \frac{1}{PH}$

$\Rightarrow \frac{1}{PE} + \frac{1}{PG} = \frac{1}{PF} + \frac{1}{PH}$

6. Show that from a set of 11 square integers one can select six numbers a^2, b^2, c^2, d^2, e^2, f^2 such that

$$a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}$$

Sol. If $I^2 \equiv x \pmod{12}$ then x can take 4-values 0, 1, 4, 9.

Now let set L_1 contains all number which are congruent mod 12.

similarly L_2, L_3,L_4 are defined

Now if out of 11-given square integers if

C-1

6 or more numbers belong to either of L_i, i = 1, 2, 3, 4 then we are through select 6 out of these numbers and these are required a^2, b^2,........,f^2.

C-2 If 5 or 4 are selected from a lot say from L_i then at least 2. must come from one of lot out of remaining 3 say L_j. Now select 4-from L_i and 2 from L_j and set a^2, b^2, d^2, e^2 from first 4 and c^2, f^2 from second set.

C-3

Now if maximum 3 are selected from one set out of '4' then selection must be 3, 3, 3,, 2 Now say L_i, L_j, L_k three are selected

where (i $\neq$ j $\neq$ k) then select 2-from L_i and set then as a^2 and d^2. Similarly select 2 from L_j and set then as b^2, e^2 and select 2-from L_k and set then as c^2, f^2. Clearly from this, given condition is satisfied.

Note that if α is maximum numbers of elements selected from a set out of four sets then $\alpha \geq 3$.